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Digital Modeling and Digital Redesign of Analog Uncertain Systems Using Genetic Algorithms

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Genetic algorithms (GAs) are utilized to find the equivalent discrete-time uncertain model of a continuous-time uncertain system for digital simulation and digital design of the continuous-time uncertain system. The developed digital interval model provides less conservative results than those obtained by the conventional bilinear transform method. Also, the global optimization searching technique provided in the GAs is used to determine the digital control law, taking into account the intersample behavior and implementation errors, for digital control of continuous-time parametric uncertain systems. The developed digitally redesigned control law is able to optimally match the states of the analogously controlled uncertain system and those of the digitally controlled sampled-data uncertain system. Moreover, it produces less conservative results than those obtained by the existing interval method. An illustrative example is included to demonstrate the proposed method.

I. Introduction

MOST practical dynamic systems, such as a helicopter,[1] a flexible structure,[2] etc., are formulated in the continuous-time uncertain settings. The uncertainties in these systems arise from unmodeled dynamics, parameter variations, sensor noises, actuator constraints, etc. For digital simulation and digital design of the continuous-time uncertain system, it is essential to find the equivalent discrete-time uncertain model from the continuous-time uncertain system. However, only a few methods are available for discretizing continuous-time state-space uncertain systems. Oppenheimer and Michel[3] first developed an interval Taylor-series approximation method to convert an autonomous continuous-time interval model to an equivalent discrete-time interval model. Also, Ezzine[4] employed a perturbation method to convert a class of continuous-time uncertain systems to their equivalent discrete-time uncertain models. Moreover, Shieh et al.[5] developed a mixed bilinear approximations method for the uncertain model conversion. Recently, Shieh et al.[6] established an interval geometric-series method with the aid of interval arithmetic to determine an enclosed equivalent discrete-time interval model. The advantage of the interval methods[3,6] is that it provides each obtained model in terms of an interval model, which contains the model sought. Nevertheless, the interval methods may give very conservative results due to the inherent conservativeness of the interval arithmetic.

For improving the properties of the continuous-time uncertain systems, there are many robust analog control design methods[7,8] available. However, it is well known that digital control provides various advantages over the analog control.[9,10] Hence, digital control of continuous-time uncertain systems has been an active research branch in recent years,[11–15] and yet, only a few methods are available for robust digital control of sampled-data, i.e., mixed continuous-time and discrete-time, uncertain systems. In the time domain, Ackermann[12] developed a parameter space design method for robust control of sampled-data uncertain systems. Also, Barmish et al.[13] used a generalized lifting technique to transform the periodic multirate discrete-time system to an equivalent single-rate linear shift-invariant discrete-time system with infinite-dimensional input and output spaces. As a result, the standard discrete-time robust control technique can be applied to synthesize such sampled-data uncertain systems. Nevertheless, the performance specifications based on the obtained discrete-time uncertain model could produce a degradation in the intersample behavior of the closed-loop sampled-data uncertain system.[11,14] A recent trend in synthesizing sampled-data uncertain systems is to develop a design technique for sampled-data uncertain systems that allow direct handling of the intersampling behavior.[11,14] Recently, with the aid of interval arithmetic, Shieh et al.[15] developed an interval digital redesign method for sampled-data uncertain systems. Basically, they consider that a robust nominal analog controller for a continuous-time uncertain system is available or that it can be predesigned. Then, they carry out the digital redesign of the available nominal analog controller, so that the states of the digitally controlled sampled-data interval system closely match those of the original continuous-time controlled uncertain system. Again, due to the use of interval arithmetic, the obtained digital interval controller may give very conservative results for the hybrid interval system.

An alternative approach to find a less conservative digital model and digital controller for a continuous-time uncertain system is the Monte Carlo statistical analysis method. Yet, application of the Monte Carlo method is computationally intensive and its rate of convergence uncertain. Hence, it is rarely used. An emerging approach to achieve the aforementioned objectives is genetic algorithm (GA), which has been successfully applied to solve global optimization searching problems.[16–20] GA is a parallel, global search technique based on natural or artificial genetic operations, which include the operators of reproduction, crossover, and mutation. It uses random choice as a tool to guide a highly exploitative search through a coding of a parameter space. GA can search many local minima in parallel, exchange information between the local minima, and thereby increase the likelihood of finding the global one. As a result, it can effectively solve complex, conflicting, mathematically difficult and constrained multiple objective problems.[21,22] In this paper, we use GAs to carry out digital modeling and digital redesign of a continuous-time uncertain system.

In the digital modeling problem, we consider that the continuous-time parametric uncertain system can be represented by a set of continuous-time nominal models generated from the perturbed system parameters via GA. Based on each nominal analog system, an equivalent nominal digital model is constructed. Then, according to a fitness function, the GA is applied to determine the desired perturbation ranges of the equivalent discrete-time interval model. With the use of an effective global searching technique provided in GA, we are able to obtain a less conservative discrete-time interval model than the model directly obtained from the interval method.

In the digital redesign problem, we predesign a robust analog controller for a continuous-time uncertain system using any existing H_2/H_∞ design method.[7,8] Then, we convert a finite set of

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continuous-time nominal models, generated from the perturbed system parameters via the GA, to an equivalent finite set of discrete-time nominal models. Based on each nominal digital model, the predetermined robust analog controller is digitally redesigned using a suboptimal digital redesign method,[23] which takes into account the intersample behavior of the closed-loop sampled-data system. Then, according to a fitness function, the GA is utilized to determine the perturbation ranges of the digitally redesigned control gains for finding the allowable implementation errors. Moreover, a global optimization searching technique provided in GA is employed to determine an implementable optimal nominal digital control law, so that the maximized integral-absolute error between the states of the analogously controlled uncertain system and those of the digitally controlled sampled-data uncertain system is minimized. The developed nominal digital control law is able to provide less conservative results than those obtained by the existing interval digital redesign method.[15]

II. Digital Modeling of Analog Uncertain Systems Using GA

Consider a continuous-time uncertain state-space system given by

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t), \qquad x_c(0) = x_{c0} \tag{1}$$

where $x_c(t) \in \Re^{n\times 1}$ is the state, $u_c(t) \in \Re^{m\times 1}$ is the input, and $(A = A_0 + \Delta A, B = B_0 + \Delta B)$ is the pair of uncertain system matrices with appropriate dimensions. In the uncertain system matrices, (A_0, B_0) are nominal system matrices and $(\Delta A, \Delta B)$ are the perturbations of the nominal system matrices with $\Delta A \equiv \{\Delta a_{ij}$ for all i and $j\}$ and $\Delta B \equiv \{\Delta b_{ij}$ for all i and $j\}$.

The uncertain system matrices (A, B) can be represented in an interval form (A^I, B^I) with $A^I = [A_0 - \Delta\tilde{A}, A_0 + \Delta\tilde{A}] = [\underline{A}, \bar{A}] \in \Im\Re^{n\times n}$, $A_0 = 0.5(\underline{A} + \bar{A})$, $\Delta\tilde{A} = 0.5(\bar{A} - \underline{A}) = \{|\Delta a_{ij}|$ for all i and $j\}$, $B^I = [B_0 - \Delta\tilde{B}, B_0 + \Delta\tilde{B}] = [\underline{B}, \bar{B}] \in \Im\Re^{n\times m}$, $B_0 = 0.5(\underline{B} + \bar{B})$, and $\Delta\tilde{B} = 0.5(\bar{B} - \underline{B}) = \{|\Delta b_{ij}|$ for all i and $j\}$. The interval analysis preliminaries are shown in the Appendix.[24–26]

For digital simulation and digital design of the continuous-time uncertain system (1), it is essential to construct an equivalent discrete-time uncertain model. The associated discrete-time uncertain model for Eq. (1) is

$$x_d(kT + T) = Gx_d(kT) + Hu_d(t), \qquad x_d(0) = x_{c0} \tag{2a}$$

where

$$G = \exp[(A_0 + \Delta A)T] \tag{2b}$$

$$H = \int_0^T \exp[(A_0 + \Delta A)\lambda](B_0 + \Delta B)\,\mathrm{d}\lambda$$

$$= (G - I_n)(A_0 + \Delta A)^{-1}(B_0 + \Delta B) \tag{2c}$$

$$u_d(t) = u_c(kT) \qquad \text{for} \qquad kT \le t < (k+1)T \tag{2d}$$

where T is the sampling period, I_n is an $(n \times n)$ identity matrix, and the piecewise-constant input $u_d(kT)$ is the output signal of a zero-order hold. Note that $e^{A_0 T}$ and $e^{\Delta AT}$, in general, are not commutative and G and H consist of nonlinear uncertainty terms in ΔA and $(\Delta A, \Delta B)$, respectively. Also, note that when $X \in \Re^{n\times n}$ is a singular matrix and $G = e^{XT}$, then the matrix-valued function $(e^{XT} - I_n)X^{-1} = (G - I_n)X^{-1}$ shall be represented as

$$\sum_{i=1}^{\infty} \frac{T(XT)^{i-1}}{i!}$$

Exact evaluation of the uncertain system matrices (G, H) in Eqs. (2) is practically impossible; hence, we determine the approximants as

$$G \cong G_0 + \Delta\tilde{G} \tag{3a}$$

$$H \cong H_0 + \Delta\tilde{H} \tag{3b}$$

where (G_0, H_0) is the pair of nominal system matrices and $(\Delta\tilde{G}, \Delta\tilde{H})$ is the pair of perturbation matrices with $\Delta\tilde{G} \equiv \{\Delta\tilde{g}_{ij}$ for all i and $j\}$ and $\Delta\tilde{H} \equiv \{\Delta\tilde{h}_{ij}$ for all i and $j\}$.

Utilizing a bilinear approximation method together with the interval arithmetic, Shieh et al.[6] developed the unstructured approximants (denoted by G_b and H_b) of G and H in Eqs. (3) as

$$G = e^{AT} \cong -I_n + 2\left(I_n - \frac{T}{2}A^I\right)^{-1} \equiv G_b \quad \text{for} \quad T < \frac{2}{\|A^I\|} \tag{4a}$$

$$H = (G - I_n)(A^I)^{-1}B^I \cong \left(I_n - \frac{T}{2}A^I\right)^{-1} B^I T \equiv H_b \quad \text{for} \quad T < \frac{2}{\|A^I\|} \tag{4b}$$

Direct use of the interval arithmetic to evaluate the approximate models in Eqs. (4) often gives very conservative results. For example, if we carry out direct computation of $(A^I)^{-1} = (A_0 + \Delta A)^{-1}$ using any regular matrix inversion method[27] together with the interval arithmetic operations, then the result obtained would be too conservative to be practically useful. Although efficient computation methods for finding $(A^I)^{-1}$ have been proposed by Hanson (see Ref. 25) and Shary[28] to provide less conservative results, their methods still give conservative results for a high-dimensional and large perturbation matrix. Moreover, due to the nature of the interval arithmetic and the inherent conservativeness of interval arithmetic operations, for example, $A^I - A^I \neq 0_n$, $(A^I)^{-1}(A^I) \neq I_n$, and $(A^I)^2 + A^I \supseteq A^I(A^I + I_n)$, the high-order approximants of G and H developed in Shieh et al.[6] may not give better results than the low-order one shown in Eqs. (4). Hence, the problem for digital modeling of a continuous-time uncertain system has not been completely solved. In this section, we take the advantages of the GA for finding the less conservative uncertain digital model (3) from the analog uncertain system (1).

Let us define the continuous-time uncertain system matrices A and B in Eq. (1) as $A = \{a^I_{ij}$ for all i and $j\} \equiv A^I_{ij} = [\underline{A}_{ij}, \bar{A}_{ij}] \in \Im\Re^{n\times n}$, where $\underline{A}_{ij} = \{\underline{a}_{ij}$ for all i and $j\} \in \Re^{n\times n}$ and $\bar{A}_{ij} = \{\bar{a}_{ij}$ for all i and $j\} \in \Re^{n\times n}$, and $B = \{b^I_{ij}$ for all i and $j\} \equiv B^I_{ij} = [\underline{B}_{ij}, \bar{B}_{ij}] \in \Im\Re^{n\times m}$, where $\underline{B}_{ij} = \{\underline{b}_{ij}$ for all i and $j\} \in \Re^{n\times m}$ and $\bar{B}_{ij} = \{\bar{b}_{ij}$ for all i and $j\} \in \Re^{n\times m}$. Also, we define two degenerate interval (real) matrices $A_r \equiv A_{rij} = \{a_{rij} \in a^I_{ij}$ for all i and $j\} \in A^I_{ij}$ and $B_r \equiv B_{rij} = \{b_{rij} \in b^I_{ij}$ for all i and $j\} \in B^I_{ij}$.

In the same manner, we define the discrete-time uncertain system matrices G and H in Eqs. (3) as $G = \{g^I_{ij}$ for all i and $j\} \equiv G^I_{ij} = [\underline{G}_{ij}, \bar{G}_{ij}]$, where $\underline{G}_{ij} = \{\underline{g}_{ij}$ for all i and $j\}$ and $\bar{G}_{ij} = \{\bar{g}_{ij}$ for all i and $j\}$, and $H = \{h^I_{ij}$ for all i and $j\} \equiv H^I_{ij} = [\underline{H}_{ij}, \bar{H}_{ij}]$, where $\underline{H}_{ij} = \{\underline{h}_{ij}$ for all i and $j\}$ and $\bar{H}_{ij} = \{\bar{h}_{ij}$ for all i and $j\}$. Also, we define two degenerate interval (real) matrices $G_r \equiv G_{rij} = \{g_{rij} \in g^I_{ij}$ for all i and $j\} \in G^I_{ij}$ and $H_r \equiv H_{rij} = \{h_{rij} \in h^I_{ij}$ for all i and $j\} \in H^I_{ij}$. The relationship between (G^I_{ij}, H^I_{ij}) and (A^I_{ij}, B^I_{ij}) for all i and j can be represented as

$$G^I_{ij} = e^{A^I_{ij}T} \tag{5a}$$

$$H^I_{ij} = \left[e^{A^I_{ij}T} - I_n\right]\left(A^I_{ij}\right)^{-1} B^I_{ij} \tag{5b}$$

The relationship between (G_{rij}, H_{rij}) and (A_{rij}, B_{rij}) for all i and j can be expressed as

$$G_{rij} = e^{A_{rij}T} \tag{6a}$$

$$H_{rij} = \left[e^{A_{rij}T} - I_n\right]A^{-1}_{rij}B_{rij} \tag{6b}$$

Note that the evaluation of $e^{A^I_{ij}T}$, $(A^I_{ij})^{-1}$, and the product of various interval matrices in Eqs. (5) directly using interval arithmetic would give very conservative results, whereas the exact evaluation of the equivalent terms in Eqs. (6) can be accomplished using real arithmetic.

Our objective is to globally search the parameter space of A^I_{ij} and B^I_{ij} for finding the extreme values for G^I_{ij} and H^I_{ij}, i.e., $(\underline{G}_{ij}, \bar{G}_{ij})$ and $(\underline{H}_{ij}, \bar{H}_{ij})$, in Eqs. (5) indirectly using the results in Eqs. (6) and GAs.

The procedure of the GA search for G and H is given as follows.

1) Arrange the interval parameters in A^I_{ij} and B^I_{ij} as an interval parameter string (denoted by Φ^I_{ab}), $\Phi^I_{ab} = [a^I_{ij}$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, n$; b^I_{ij} for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n]$ and encode the lower and upper bound values of each interval parameter as binary strings of 0s and 1s with a specified bit length.

2) Generate the initial population with randomly selected N sets of encoded parameter strings (denoted by $\Phi^{(l)}_{eab}$ for $l = 1, 2, \ldots, N$). Also, specify the number of the generation.

3) Decode each parameter string in the population as a real parameter string (denoted by $\Phi^{(l)}_{rab}$), $\Phi^{(l)}_{rab} = [a^{(l)}_{rij}$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, n$; $b^{(l)}_{rij}$ for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n]$. Utilize each real parameter string in the population to reconstruct its associated real parameter matrices $A^{(l)}_{rij} = \{a^{(l)}_{rij}$ for all i and $j\}$ and $B^{(l)}_{rij} = \{b^{(l)}_{rij}$ for all i and $j\}$ and compute the equivalent discrete-time real models from Eqs. (6) as $G^{(l)}_{rij} = \{g^{(l)}_{rij}$ for all i and $j\}$ and $H^{(l)}_{rij} = \{h^{(l)}_{rij}$ for all i and $j\}$.

4) For a separate search of each interval element in g^I_{ij} and h^I_{ij}, select a specific interval parameter (denoted by $x^{I(l)}_{ij}$) of interest from $g^{I(l)}_{ij}$ and $h^{I(l)}_{ij}$ and determine the lower (upper) bound value of $x^{I(l)}_{ij}$, i.e., $\underline{x}^{(l)}_{ij}$ $(\bar{x}^{(l)}_{ij})$, using the obtained specific real parameter (denoted by $x^{(l)}_{rij}$) from $g^{(l)}_{rij}$ and $h^{(l)}_{rij}$ for $l = 1, 2, \ldots, N$. Next, calculate the fitness value of the specific real parameter $x^{(l)}_{rij}$ for each associated real parameter string $(\Phi^{(l)}_{rab})$ in the population using the following fitness functions:

$$\underline{f}\left(\Phi^{(l)}_{rab}\right) = \begin{cases} \dfrac{1}{\left(x^{(l)}_{rij} - \underline{x}^{(l)}_{ij}\right) + 1} & \text{if} \quad x^{(l)}_{rij} \geq \underline{x}^{(l)}_{ij} \\ \left(\underline{x}^{(l)}_{ij} - x^{(l)}_{rij}\right) + 1 & \text{if} \quad x^{(l)}_{rij} < \underline{x}^{(l)}_{ij} \end{cases} \quad \text{for} \quad l = 1, 2, \ldots, N \tag{7a}$$

and

$$\bar{f}\left(\Phi^{(l)}_{rab}\right) = \begin{cases} \left(x^{(l)}_{rij} - \bar{x}^{(l)}_{ij}\right) + 1 & \text{if} \quad x^{(l)}_{rij} \geq \bar{x}^{(l)}_{ij} \\ \dfrac{1}{\left(\bar{x}^{(l)}_{ij} - x^{(l)}_{rij}\right) + 1} & \text{if} \quad x^{(l)}_{rij} < \bar{x}^{(l)}_{ij} \end{cases} \quad \text{for} \quad l = 1, 2, \ldots, N \tag{7b}$$

Based on the calculated fitness values $\underline{f}(\Phi^{(l)}_{rab})[\bar{f}(\Phi^{(l)}_{rab})]$ of the $x^{(l)}_{rij}$ for $l = 1, 2, \ldots, N$, reproduce N sets of high-quality encoded parameter strings $(\Phi^{(l)}_{eab})$ to form a new population for the associated $\underline{x}^{(l)}_{ij}(\bar{x}^{(l)}_{ij})$.

5) Specify crossover rate and mutation rate and then perform the crossover and mutation operations to the obtained high-quality strings to reproduce N sets of new offspring strings (a next generation) for the associated $\underline{x}^{(l)}_{ij}(\bar{x}^{(l)}_{ij})$.

6) Calculate the fitness value $\underline{f}(\Phi^{(l)}_{rab})[\bar{f}(\Phi^{(l)}_{rab})]$ for each of the newly reproduced offspring string. If the fitness value cannot be improved further and/or the allowable generation is achieved, determine the desired lower (upper) bound value for the $x^{I(l)}_{ij}$ based on the newly reproduced offspring strings for the associated $\underline{x}^{(l)}_{ij}(\bar{x}^{(l)}_{ij})$. Otherwise, go to step 3 and continue until the desired extreme values for all interval parameters (g^I_{ij} and h^I_{ij}) are found.

Remark 1. We made a separate search for each interval element in g^I_{ij} and h^I_{ij} instead of a simultaneous search for the entire set of interval parameters in g^I_{ij} and h^I_{ij}. Hence, better extreme values for each element in g^I_{ij} and h^I_{ij} can be obtained. A parallel search method can be utilized to simultaneously search the extreme values for individual g^I_{ij} and h^I_{ij}. Also, based on Eqs. (2b) and (2c), we observe that the interval matrix G depends on ΔA only and the other interval matrix H involves both ΔA and ΔB. Hence, we can determine the interval matrices G and H independently via GAs for obtaining better results.

III. Digital Redesign of Analog Uncertain Systems Using GAs

Rewrite the interval state equation (1) as

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t), \qquad x_c(0) = x_{c0} \tag{8a}$$

and the constant gain control law to be

$$u_c(t) = -K_{cr}x_c(t) + E_{cr}r(t) \tag{8b}$$

where $K_{cr} \in \Re^{m\times n}$ and $E_{cr} \in \Re^{m\times m}$. It is assumed that the constant gain control law (8b) is available or that it can be designed via any existing H_2/H_∞ control theory.[7,8]

The closed-loop interval system of Eq. (8) is

$$\dot{x}_c(t) = A_c x_c(t) + BE_{cr}r(t) \tag{9}$$

where $A_c \equiv (A - BK_{cr})$. If $r(t)$ is a piecewise-constant input, defined as

$$r(t) = r(kT) \qquad \text{for} \qquad kT \leq t < (k+1)T \tag{10a}$$

then the discretized solution of $x_c(t)$ at $t = kT + iT_N$ (where $T_N \equiv T/N$, N is an integer, and $i = 1, 2, \ldots$) in Eq. (9) can be solved as

$$x_c(kT + iT_N) = G^{(i)}_c x_c(kT) + H^{(i)}_c E_{cr}r(kT) \tag{10b}$$

where $G^{(i)}_c \equiv (e^{A_cT_N})^i$ and $H^{(i)}_c \equiv (G^{(i)}_c - I_n)A_c^{-1}B$. When $i = N$, the discrete-time state equation (10b) becomes

$$x_c(kT + T) = G_c x_c(kT) + H_c E_{cr}r(kT) \tag{11}$$

where $G_c \equiv e^{A_cT}$ and $H_c \equiv (G_c - I_n)A_c^{-1}B$.

Also, let the interval state equation (8a) with a piecewise-constant input $u_d(t)$ be

$$\dot{x}_d(t) = Ax_d(t) + Bu_d(t), \qquad x_d(0) = x_{c0} \tag{12a}$$

$$u_d(t) = u_d(kT) = -K_{dr}x_d(kT) + E_{dr}r(kT) \quad \text{for} \quad kT \leq t < (k+1)T \tag{12b}$$

where $u_d(kT)$ is the output signal of a zero-order hold, $K_{dr} \in \Re^{m\times n}$ and $E_{dr} \in \Re^{m\times m}$ are the digital gains, and $r(kT)$ is shown in Eq. (10a). The resulting closed-loop hybrid interval system becomes

$$\dot{x}_d(t) = Ax_d(t) - BK_{dr}x_d(kT) + BE_{dr}r(kT) \tag{13}$$

The corresponding interval discrete-time closed-loop system in Eq. (13) evaluated at $t = kT + iT_N$ is

$$x_d(kT + iT_N) = \left(G^{(i)} - H^{(i)}K_{dr}\right)x_d(kT) + H^{(i)}E_{dr}r(kT) \tag{14a}$$

where $G^{(i)} \equiv (e^{AT_N})^i$ and $H^{(i)} \equiv (G^{(i)} - I_n)A^{-1}B$ for $i = 1, 2, \ldots$.

When $i = N$, we have

$$x_d(kT + T) = (G - HK_{dr})x_d(kT) + HE_{dr}r(kT) \tag{14b}$$

where $G \equiv e^{AT}$ and $H \equiv (G - I_n)A^{-1}B$.

It is desired to find the digital control law in Eq. (12b) from the analog control law in Eq. (8b) such that the state $x_d(t)$ of the digitally controlled sampled-data uncertain system in Eq. (13) is able to optimally match the state $x_c(t)$ of the analogously controlled uncertain system in Eq. (9). The process of the aforementioned control law conversion is called state-matching digital redesign.[9,15,23]

We extend the suboptimal matching digital redesign method[23] developed for a nominal system (1), i.e., $A \equiv A_0$, $B \equiv B_0$, to an uncertain system (1) for finding the robust nominal digital control law (12b). The suboptimal digital redesign method[23] can be briefly described as follows.

Evaluating the closed-loop states $x_c(t)$ in Eq. (8) with $A := A_r$ and $B := B_r$ at $t = kT + T$ yields

$$x_c(kT+T) = G_r x_c(kT) + \int_{kT}^{kT+T} \exp[A_r(kT+T-\lambda)]B_r u_c(\lambda)\,\mathrm{d}\lambda \tag{15a}$$

where $G_r = e^{A_r T} \in G$, $A_r \in A$, and $B_r \in B$.

Based on the law of mean from integral calculus,[29] there exists a piecewise-constant function $u_c(t_v)$, where $t_v \equiv kT + vT$ and $0 \le v \le 1$, such that the integral term in Eq. (15a) can be represented as

$$\int_{kT}^{kT+T} \exp[A_r(kT+T-\lambda)]B_r u_c(\lambda)\,\mathrm{d}\lambda = \int_{kT}^{kT+T} \exp[A_r(kT+T-\lambda)]B_r\,\mathrm{d}\lambda u_c(t_v) = H_r u_c(t_v) \tag{15b}$$

where $H_r = (G_r - I_n)A_r^{-1}B_r$.

If $u_c(t_v)$ in Eq. (15b) can be found, the closed-loop states $x_c(t)$ in Eq. (8) with $A := A_r$ and $B := B_r$ and $x_d(t)$ in Eq. (12) with $A := A_r$ and $B := B_r$ at $t = t_v$ can be evaluated as

$$x_c(t_v) = \exp[A_r(t_v - kT)]x_c(kT) + \int_{kT}^{t_v} \exp[A_r(t_v-\lambda)]B_r\,\mathrm{d}\lambda u_c(t_v) = G_r^{(v)}x_c(kT) + H_r^{(v)}u_c(t_v) \tag{16a}$$

and

$$x_d(t_v) = \exp[A_r(t_v - kT)]x_d(kT) + \int_{kT}^{t_v} \exp[A_r(t_v-\lambda)]B_r\,\mathrm{d}\lambda u_d(kT) = G_r^{(v)}x_d(kT) + H_r^{(v)}u_d(kT) \tag{16b}$$

where $G_r^{(v)} = e^{A_r T v}$, $H_r^{(v)} = [G_r^{(v)} - I_n]A_r^{-1}B_r$, and $u_d(kT)$ is the desired control law to be determined.

For matching the state $x_c(t_v)$ in Eq. (16a) with the state $x_d(t_v)$ in Eq. (16b), we make $x_c(t_v) = x_d(t_v)$. As a result, the $u_c(t)$ in Eqs. (8b) and (16a) at $t = t_v$ becomes

$$u_c(t_v) = -K_{cr}x_c(t_v) + E_{cr}r(t_v) = -K_{cr}x_d(t_v) + E_{cr}r(t_v) = u_d(kT) \tag{17}$$

Substituting Eq. (16b) into Eq. (17) and solving for $u_d(kT)$ gives

$$u_d(kT) = -K_{dr}x_d(kT) + E_{dr}r(kT) \tag{18}$$

where

$$K_{dr} = \left(I_m + K_{cr}H_r^{(v)}\right)^{-1}K_{cr}G_r^{(v)}, \qquad E_{dr} = \left(I_m + K_{cr}H_r^{(v)}\right)^{-1}E_{cr}$$

The choice of the tuning parameter v depends on the specific sampling period T and the desired closeness of the state $x_d(t)$ of the digitally redesigned closed-loop system (13) with $A := A_r$ and $B := B_r$ and the original closed-loop system (9) with $A := A_r$ and $B := B_r$. The integral absolute-error criterion over the outputs

$$J(v) = \sum_{i=1}^{n}\left(\int_0^{t_f} |x_{ci}(t) - x_{di}(t)|\,\mathrm{d}t\right) \cong \sum_{i=1}^{n}\left(\sum_{j=1}^{n_f} |x_{ci}(jT_f) - x_{di}(jT_f)|T_f\right) \tag{19}$$

was utilized for the selection of the tuning parameter v, where t_f is the finite time of interest, $T_f = t_f/n_f$ with a sufficiently large integer n_f, and $x_{ci}(t)$ and $x_{di}(t)$ are the ith state variables of the closed-loop state vectors $x_c(t)$ and $x_d(t)$ in Eqs. (9) and (13) for $A := A_r$ and $B := B_r$ with the digitally redesigned control gains in Eq. (18), respectively. For practical computation, the integral term in Eq. (19) can be approximately evaluated using the discretized $x_{ci}(t)$ and $x_{di}(t)$ in Eqs. (10b) and (14a) with $A := A_r$ and $B := B_r$, respectively.

Remark 2. The selection of v in Eq. (19) involves the intersample states $x_c(t)$ and $x_d(t)$. Hence, the digital control law developed in Eq. (18) captures the intersample behavior.

By using the available analog control law in Eq. (8b) with the nominal gains (K_{cr}, E_{cr}), we globally search the parameter space of A_{ij}^I and B_{ij}^I for finding the following digitally redesigned interval control law:

$$u_d(kT) = -K_d x_d(kT) + E_d r(kT) \tag{20}$$

where the interval control gains are denoted by $K_d \equiv K_{dij}^I = \{k_{dij}^I$ for all i and $j\}$ and $E_d \equiv E_{dij}^I = \{e_{dij}^I$ for all i and $j\}$. The aforementioned digital redesign technique and GAs are employed for finding the extreme values for K_{dij}^I and E_{dij}^I, i.e., $(\underline{K}_{dij}^I, \bar{K}_{dij}^I)$ and $(\underline{E}_{dij}^I, \bar{E}_{dij}^I)$, in Eq. (20). Then, to determine an implementable nominal digital control law from the obtained interval control law in Eq. (20), we use the global optimization searching technique provided in GAs to search the parameter space of K_{dij}^I and E_{dij}^I together with those of A_{ij}^I and B_{ij}^I for finding the implementable nominal control gains for K_d and E_d in Eq. (20) and the allowable implementation errors for the control gains.

The procedure of GA search for K_d and E_d is given as follows.

1) Repeat steps 1–3 shown in the procedure of GA search in Sec. II to construct the continuous-time real parameter matrices $(A_{rij}^{(l)}, B_{rij}^{(l)})$ and the discrete-time ones $(G_{rij}^{(l)}, H_{rij}^{(l)})$.

2) Utilizing the available analog control gains (K_{cr}, E_{cr}) and the obtained system matrices $(A_{rij}^{(l)}, B_{rij}^{(l)})$ and $(G_{rij}^{(l)}, H_{rij}^{(l)})$, find the digitally redesigned real gains (denoted by $K_{drij}^{(l)} = \{k_{drij}^{(l)}$ for all i and $j\}$ and $E_{drij}^{(l)} = \{e_{drij}^{(l)}$ for all i and $j\}$) for $l = 1, 2, \ldots, N$.

3) For a separate search of each interval element in k_{dij}^I and e_{dij}^I, select a specific interval parameter (denoted by $x_{ij}^{I(l)}$) of interest from $k_{dij}^{I(l)}$ and $e_{dij}^{I(l)}$ and then determine the lower (upper) bound value of $x_{ij}^{I(l)}$, i.e., $\underline{x}_{ij}^{(l)}$ $(\bar{x}_{ij}^{(l)})$, using the obtained specific real parameter (denoted by $x_{rij}^{(l)}$) from $k_{drij}^{(l)}$ and $e_{drij}^{(l)}$ for $l = 1, 2, \ldots, N$.

4) Calculate the fitness value of the specific real gain $x_{rij}^{(l)}$ for each associated encoded parameter string $(\Phi_{eab}^{(l)})$ in the population using the associated fitness function shown in Eq. (7). Then, randomly create N sets of high-quality encoded parameter strings $(\Phi_{eab}^{(l)})$ to form a new population.

5) Apply the GA's operators to the newly created population and repeat the procedure until the desired extreme values for all interval gains (k_{dij}^I, e_{dij}^I) are found.

For practical applications, we need to implement only one digital control law, with real gains (denoted by K_{d0} and E_{d0}), which will provide suitable performance for every plant with the uncertain parameter range. In Ref. 15, it was suggested that the implementable nominal digital gains (denoted by $\tilde{K}_{d0}$ and $\tilde{E}_{d0}$) can be obtained by taking the mean values of the extreme gains, i.e., $\tilde{K}_{d0} = 0.5(\underline{K}_d + \bar{K}_d)$ and $\tilde{E}_{d0} = 0.5(\underline{E}_d + \bar{E}_d)$. Also, the allowable perturbation gains (denoted by $\Delta\tilde{K}_d$ and $\Delta\tilde{E}_d$) are $\Delta\tilde{K}_d = 0.5(\bar{K}_d - \underline{K}_d)$ and $\Delta\tilde{E}_d = 0.5(\bar{E}_d - \underline{E}_d)$. The selection of the nominal gains $(\tilde{K}_{d0}, \tilde{E}_{d0})$ in Ref. 15 is quite arbitrary. In other words, the resulting controller may not give the best performance for the closed-loop sampled-data uncertain system.

Based on the obtained interval control gains $K_d(=K_{dij}^I)$ and $E_d(=E_{dij}^I)$ and the original continuous-time interval system parameters $A(=A_{ij}^I)$ and $B(=B_{ij}^I)$, we propose a global optimization searching technique for finding the best nominal digital gains (K_{d0}, E_{d0}) as follows.

Let the nominal digital control law be

$$u_d(kT) = -K_{d0}x_d(kT) + E_{d0}r(kT) \tag{21}$$

The problem is to find $u_d(kT)$ in Eq. (21) such that the maximized objective function, $J(K_{d0}, E_{d0})$, is minimized, i.e., min[max $J(K_{d0}, E_{d0})$].

The integral absolute-error criterion is

$$J(K_{d0}, E_{d0}) = \sum_{i=1}^{n}\left(\int_0^{t_f} |x_{ci}(t) - x_{di}(t)|\,\mathrm{d}t\right) \cong \sum_{i=1}^{n}\left(\sum_{j=1}^{n_f} |x_{ci}(jT_f) - x_{di}(jT_f)|T_f\right) \tag{22a}$$

$$\dot{x}_c(t) = (A_r - B_r K_{cr})x_c(t) + B_r E_{cr} r(t) \tag{22b}$$

$$\dot{x}_d(t) = A_r x_d(t) + B_r[-K_{d0}x_d(kT) + E_{d0}r(kT)] \tag{22c}$$

where $A_r \in A$, $B_r \in B$, $K_{d0} \in K_d$, and $E_{d0} \in E_d$.

The procedure of the GA search for K_{d0} and E_{d0} is given as follows.

1) Generate an initial population with randomly selected M sets of the encoded control gain strings (denoted by $\Theta^{(q)}_{ek_de_d}$ for $q = 1, 2, \ldots, M$). Decode $\Theta^{(q)}_{ek_de_d}$ to reconstruct the real control gains $K^{(q)}_{drij}$ and $E^{(q)}_{drij}$, which belong to K^I_{dij} and E^I_{dij}, respectively.

2) For each set of $\Theta^{(q)}_{ek_de_d}$ in the population, generate another population with randomly selected N sets of the encoded system parameter strings (denoted by $\Phi^{(l)}_{eab}$ for $l = 1, 2, \ldots, N$). Decode the $\Phi^{(l)}_{eab}$ to reconstruct the real parameter matrices $(A^{(l)}_{rij}, B^{(l)}_{rij})$, which belong to (A^I_{ij}, B^I_{ij}), and compute the real parameter matrices $(G^{(l)}_{rij}, H^{(l)}_{rij})$ from Eqs. (6). Utilize the available analog control gain (K_{cr}, E_{cr}), the reconstructed digital control gains $(K^{(q)}_{drij}, E^{(q)}_{drij})$, and the real system matrices $(A^{(l)}_{rij}, B^{(l)}_{rij})$ and $(G^{(l)}_{rij}, H^{(l)}_{rij})$ to compute the performance indices (denoted by $J^{(q,l)}_{k_de_d}$, for $l = 1, 2, \ldots, N$ and a fixed value q) from Eq. (22). Then, calculate the fitness function shown in Eq. (7b), using the computed $J^{(q,l)}_{k_de_d}$, for finding the maximum value of $J^{(q,l)}_{k_de_d}$. Utilize the computed fitness values to reproduce N sets of high-quality encoded system parameter strings $(\Phi^{(l)}_{eab})$ to create a new population. Apply the GA's operators to the newly created population until the maximum value of $J^{(q,l)}_{k_de_d}$ associated with the set of $\Theta^{(q)}_{ek_de_d}$ is found. Then, repeat the procedure until the maximum values of $J^{(q,l)}_{k_de_d}$ associated with $\Theta^{(q)}_{ek_de_d}$ for $q = 1, 2, \ldots, M$ are found.

3) For finding the minimum value of the max $J^{(q,l)}_{k_de_d}$, calculate the fitness values from the fitness function shown in Eq. (7a) using the obtained max $J^{(q,l)}_{k_de_d}$ for $q = 1, 2, \ldots, M$. Then, use the calculated fitness values to reproduce M sets of high-quality encoded control gain strings ($\Theta^{(q)}_{ek_de_d}$ for $q = 1, 2, \ldots, M$), and use them to reconstruct the real control gains $K^{(q)}_{drij}$ and $E^{(q)}_{drij}$. If the fitness value cannot be improved further and/or the allowable generation is achieved, determine the desired nominal control gains $K^{(q)}_{drij}$, i.e., K_{d0}, and $E^{(q)}_{drij}$, i.e., E_{d0}. Otherwise, go to step 2.

For practical implementation of the digital gains (K_{d0}, E_{d0}) in the nominal digital control law in Eq. (21), the implementation errors[10] such as computer wordlength, electronic noise, etc., might occur. The allowable perturbation gains (denoted by ΔK_d and ΔE_d) are $\Delta K_d = K_d - K_{d0}$ and $\Delta E_d = E_d - E_{d0}$.

The digitally redesigned sampled-data uncertain system with the nominal digital control law (21) becomes

$$\dot{x}_d(t) = Ax_d(t) + Bu_d(kT) \tag{23a}$$

$$u_d(kT) = -K_{d0}x_d(kT) + E_{d0}r(kT) \tag{23b}$$

The worst-case robust stability of the digitally redesigned sampled-data interval system in Eq. (23) can be determined using the worst-case discrete-time interval system matrix (G and H) obtained in Sec. II and the interval digital control law in Eq. (20), i.e., $G - HK_d$. In other words, any existing robust stability test method[30] can be applied to check the stability of the digitally redesigned system matrix $G - HK_d$. Whenever the digitally redesigned system is unstable, the digital control law need to be redesigned using a suitably small sampling period. A bisection search is suggested to find a suitable sampling period.

IV. Illustrative Example

The dynamics of a helicopter in a vertical plane for an airspeed range of 60–170 kn are given by Narendra and Tripathi.[1] There are four state variables: x_1 = horizontal velocity (knot/second), x_2 = vertical velocity (knot/second), x_3 = pitch rate (degree/second), x_4 = pitch angle (degree); and two control variables: u_1 = collective pitch control and u_2 = longitudinal cyclic pitch control. In the airspeed range of 60–170 kn, significant changes occur only in the elements a^I_{32} and a^I_{34} of A and b^I_{21} of B.

The aforementioned uncertain system can be represented in Eq. (1) with the nominal and perturbation matrices as

$$A_0 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.7070 & 1.3229 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.0447 & -7.5922 \\ -5.5200 & 4.9900 \\ 0 & 0 \end{bmatrix}$$

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \pm 0.2192 & 0 & \pm 1.2031 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Delta B = \begin{bmatrix} 0 & 0 \\ \pm 2.0673 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{24}$$

It is desired to find the equivalent discrete-time uncertain model of the continuous-time uncertain system in Eq. (1), with the uncertain system matrices in Eq. (24), for digital simulation and digital design of the uncertain system in Eq. (1). Also, it is desired to find a digital controller for robust control of the sampled-data uncertain system taking into account the intersample behavior and implementation errors.

We use the GA described in Sec. II to search (G, H), denoted by (G_g, H_g), respectively. Here, the sampling period $T = 0.2$ s, the population size is chosen as 100, and the crossover rate and mutation rate are 0.6 and 0.05, respectively. Each entry in A and B is encoded by an eight-bit-long binary string. After 50 generations, we obtain the following results:

$$G_g = \begin{pmatrix} [0.9927 \quad 0.9927] & [0.0048 \quad 0.0049] & [-0.0056 \quad -0.0050] & [-0.0934 \quad -0.0928] \\ [0.0082 \quad 0.0082] & [0.8148 \quad 0.8168] & [-0.0721 \quad -0.0713] & [-0.7409 \quad -0.7287] \\ [0.0187 \quad 0.0194] & [0.0112 \quad 0.0865] & [0.8679 \quad 0.9142] & [-0.0148 \quad 0.4733] \\ [0.0019 \quad 0.0020] & [0.0012 \quad 0.0091] & [0.1865 \quad 0.1896] & [0.9998 \quad 1.0482] \end{pmatrix}$$

$$H_g = \begin{pmatrix} [0.0899 \quad 0.0920] & [0.0301 \quad 0.0301] \\ [0.2042 \quad 0.9531] & [-1.3991 \quad -1.3982] \\ [-1.0447 \quad -0.9836] & [0.8627 \quad 0.9376] \\ [-0.1061 \quad -0.1022] & [0.0906 \quad 0.0955] \end{pmatrix} \tag{25}$$

The existing bilinear approximants in Eqs. (4a) and (4b) give

$$G_b = \begin{pmatrix} [0.9927 \quad 0.9927] & [0.0048 \quad 0.0049] & [-0.0053 \quad -0.0053] & [-0.0934 \quad -0.0928] \\ [0.0082 \quad 0.0082] & [0.8148 \quad 0.8168] & [-0.0723 \quad -0.0711] & [-0.7412 \quad -0.7283] \\ [0.0186 \quad 0.0194] & [0.0101 \quad 0.0871] & [0.8680 \quad 0.9140] & [-0.0198 \quad 0.4745] \\ [0.0019 \quad 0.0020] & [0.0012 \quad 0.0090] & [0.1857 \quad 0.1904] & [0.9987 \quad 1.0491] \end{pmatrix}$$

$$H_b = \begin{pmatrix} [0.0899 \quad 0.0920] & [0.0300 \quad 0.0301] \\ [0.2035 \quad 0.9537] & [-1.3997 \quad -1.3976] \\ [-1.0643 \quad -0.9792] & [0.8601 \quad 0.9398] \\ [-0.1087 \quad -0.1007] & [0.0890 \quad 0.0971] \end{pmatrix} \tag{26}$$

To make quantitative comparison between our approximate models (G_g, H_g) in Eqs. (25) and the existing bilinear approximants (G_b, H_b) in Eqs. (26), we apply a partition method, though costly, to exhaustively search for the reference approximants [denoted by (G_p, H_p)]. (G_p, H_p) can be considered as exact models. The partition method can be described as follows.

From Eq. (24), we observe that the uncertain system matrix A consists of two parametric uncertainties, whereas the uncertain input vector B contains one parametric uncertainty. Hence, we partition each of the three perturbed parameter intervals into 2^8 equal width segments, which result in $(2^8)^3$ combinations of real parameters. Then, we carry out the digital model conversions of $(2^8)^3$ continuous-time nominal models and determine the extreme points in each entry of the digital models to form the desired reference interval models (G_p, H_p) as follows:

$$G_p = \begin{pmatrix} [0.9927 \quad 0.9927] & [0.0048 \quad 0.0049] & [-0.0056 \quad -0.0050] & [-0.0934 \quad -0.0928] \\ [0.0082 \quad 0.0082] & [0.8148 \quad 0.8168] & [-0.0721 \quad -0.0713] & [-0.7409 \quad -0.7287] \\ [0.0187 \quad 0.0194] & [0.0112 \quad 0.0865] & [0.8679 \quad 0.9142] & [-0.0148 \quad 0.4733] \\ [0.0019 \quad 0.0020] & [0.0012 \quad 0.0091] & [0.1865 \quad 0.1896] & [0.9997 \quad 1.0482] \end{pmatrix}$$

$$H_p = \begin{pmatrix} [0.0899 \quad 0.0920] & [0.0301 \quad 0.0301] \\ [0.2042 \quad 0.9531] & [-1.3991 \quad -1.3982] \\ [-1.0448 \quad -0.9827] & [0.8627 \quad 0.9376] \\ [-0.1061 \quad -0.1022] & [0.0906 \quad 0.0955] \end{pmatrix} \tag{27}$$

The computation time for the results in Eqs. (25) via the genetic algorithm method is less than 30 min, whereas the partition method takes more than 10 h to obtain the results in Eqs. (27). Based on the simulation results shown in Eqs. (25–27), we make the quantitative comparisons as follows:

$$\|G_g\| = 1.6379 < \|G_p\| = 1.6380 < \|G_b\| = 1.6385$$

$$\|H_g\| = 2.3522 < \|H_p\| = 2.3523 < \|H_b\| = 2.3533$$

Also,

$$\|\underline{G}_g - \underline{G}_p\| = 3.8 \times 10^{-5} < \|\underline{G}_b - \underline{G}_p\| = 0.0062$$

$$\|\bar{G}_g - \bar{G}_p\| = 2.6 \times 10^{-5} < \|\bar{G}_b - \bar{G}_p\| = 0.0019$$

$$\|\underline{H}_g - \underline{H}_p\| = 9.8 \times 10^{-5} < \|\underline{H}_b - \underline{H}_p\| = 0.0220$$

$$\|\bar{H}_g - \bar{H}_p\| = 8.8 \times 10^{-4} < \|\bar{H}_b - \bar{H}_p\| = 0.0057$$

Hence, based on these comparison results, we conclude that the proposed approximate models (G_g, H_g) are close to the reference models (G_p, H_p). Also, the proposed approximate models (G_g, H_g) are less conservative than the existing bilinear approximants (G_b, H_b).

To solve the digital redesign problem for the continuous-time uncertain system (1), we utilize the H_∞ controller design method[7,15] to predesign the analog H_∞ controller $u_c(t)$ in Eq. (8b) with the nominal gains as

$$K_{cr} = \begin{bmatrix} 2.7957 & -0.0160 & -0.7277 & -2.5223 \\ -1.4748 & -1.2515 & -0.2618 & 1.3956 \end{bmatrix}$$

$$E_{cr} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{28}$$

Applying the proposed digital redesign method together with the GAs in Sec. III yields the robust interval digital control law in Eq. (20) with the nominal gains as

$$K_{d0} = \begin{bmatrix} 1.6529 & -0.1707 & -0.6421 & -1.6731 \\ -0.6928 & -0.6778 & -0.1039 & 0.8342 \end{bmatrix}$$

$$E_{d0} = \begin{bmatrix} 0.6524 & 0.1283 \\ 0.1134 & 0.5749 \end{bmatrix} \tag{29}$$

and the interval gains as

$$K_d = \begin{pmatrix} [1.5751 \quad 1.7188] & [-0.1828 \quad -0.1516] & [-0.6508 \quad -0.5394] & [-1.6983 \quad -1.4419] \\ [-0.7464 \quad 0.2583] & [-0.6901 \quad -0.6271] & [-0.2613 \quad -0.0737] & [0.5003 \quad 1.0100] \end{pmatrix}$$

$$E_d = \begin{pmatrix} [0.5857 \quad 0.6840] & [0.1219 \quad 0.1522] \\ [0.0299 \quad 0.2235] & [0.5582 \quad 0.5977] \end{pmatrix} \tag{30}$$

The allowable implementation errors for the control gains are $\Delta K_d = K_d - K_{d0}$ and $\Delta E_d = E_d - E_{d0}$.

Note that, for finding the nominal digital control gains ($K_{drij}^{(l)}$ and $E_{drij}^{(l)}$) from the nominal analog control gains ($K_{crij}^{(l)}$ and $E_{crij}^{(l)}$) via the aforementioned suboptimal digital redesign method, the range of the tuning parameter v in Eq. (19) with $t_f = 6$ s and $r(t) = 1$ for $t \geq 0$ was found to be [0.50, 0.51], and the extreme value in Eq. (22a), $\min[\max J(K_{drij}^{(l)}, E_{drij}^{(l)})]$, is 1.1248.

The digitally redesigned discrete-time closed-loop interval system is

$$x_d(kT + T) = (G_g - H_g K_d)x_d(kT) + H_g E_d r(kT) \tag{31a}$$

where the closed-loop system matrix is

$$G_g - H_g K_d = \begin{pmatrix} [0.8615 \quad 0.8650] & [0.0405 \quad 0.0410] & [0.0553 \quad 0.0572] & [0.0319 \quad 0.0360] \\ [-2.5366 \quad -1.2981] & [-0.0987 \quad 0.0318] & [-0.0863 \quad 0.3955] & [0.7671 \quad 2.0332] \\ [2.2407 \quad 2.3958] & [0.4176 \quad 0.5543] & [0.2867 \quad 0.3806] & [-2.5450 \quad -1.8905] \\ [0.2336 \quad 0.2435] & [0.0363 \quad 0.0445] & [0.1278 \quad 0.1340] & [0.7426 \quad 0.8017] \end{pmatrix} \tag{31b}$$

The sufficient condition provided in the robust test methods[30] has been applied to check the stability of the interval system matrix in Eq. (31). The stability test results are inconclusive for this example although the uncertain system matrices evaluated at some fixed extreme parameter values are stable. Both the genetic algorithm and Monte Carlo evaluation method have been utilized to estimate the eigenvalues of the closed-loop uncertain system matrix. The resulting worst-case eigenvalue is 0.9247, which is less than one. Hence, the digitally redesigned closed-loop uncertain system seems to be robustly stable for the sampling period $T = 0.2$ s.

To compare the results developed in this paper with those of the existing method,[15] we determine the digitally redesigned control law from the nominal control law in Eq. (8b) with the nominal gains in Eq. (28) via the existing method[15] as

$$u_d(kT) = -\tilde{K}_{d0} x_d(kT) + \tilde{E}_{d0} r(kT) \tag{32}$$

where

$$\tilde{K}_{d0} = \begin{bmatrix} 1.4541 & -0.2114 & -0.6113 & -1.4148 \\ -0.3506 & -0.5950 & -0.1564 & 0.65555 \end{bmatrix}$$

$$\tilde{E}_{d0} = \begin{bmatrix} 0.6116 & 0.1639 \\ 0.1548 & 0.5290 \end{bmatrix}$$

The associated digitally redesigned sampled-data uncertain system with the nominal digital control law in Eq. (32) is

$$\dot{\tilde{x}}_d(t) = A\tilde{x}_d(t) + Bu_d(kT) \tag{33a}$$

$$u_d(kT) = -\tilde{K}_{d0}\tilde{x}_d(kT) + \tilde{E}_{d0} r(kT) \tag{33b}$$

To compare the performances of the digitally redesigned controllers in Eqs. (23b) and (33b), we define the performance indices as

$$J(K_{d0}, E_{d0}) = \sum_{i=1}^{n} \left(\int_0^{t_f} |x_{ci}(t) - x_{di}(t)| \, \mathrm{d}t \right) \tag{34a}$$

and

$$J(\tilde{K}_{d0}, \tilde{E}_{d0}) = \sum_{i=1}^{n} \left(\int_0^{t_f} |x_{ci}(t) - \tilde{x}_{di}(t)| \, \mathrm{d}t \right) \tag{34b}$$

where

$$\dot{x}_c(t) = (A - BK_{cr})x_c(t) + BE_{cr} r(t)$$

$$\dot{x}_d(t) = Ax_d(t) + B[-K_{d0}x_d(kT) + E_{d0} r(kT)]$$

$$\dot{\tilde{x}}_d(t) = A\tilde{x}_d(t) + B[-\tilde{K}_{d0}\tilde{x}_d(kT) + \tilde{E}_{d0} r(kT)]$$

The worst-case performance indices for $t_f = 6$ s and $r(t) = 1$ for $t \geq 0$ are compared as follows:

$$\max J(K_{d0}, E_{d0}) = 1.1248 < \max J(\tilde{K}_{d0}, \tilde{E}_{d0}) = 1.3679 \tag{35}$$

From the result [Eq. (35)], we conclude that the proposed control law produces better performance and less conservative results than the existing control law.

V. Conclusions

Based on GAs, a new approach for digital modeling and digital design of a continuous-time parametric uncertain system is presented.

GAs, with a proper fitness function, are utilized to effectively search the lower and upper bounds of the interval parameters for the discrete-time parametric uncertain system from the available continuous-time parametric uncertain system. The developed discrete-time uncertain model provides less conservative results than those provided by conventional techniques.

Also, the global optimization searching technique, provided in the GAs, is employed to optimally search the lower and upper bounds of the digital interval control gains from the available analog control gains for digital control of continuous-time parametric uncertain systems. The developed digital control law, which captures intersample behavior and takes into account implementation errors, is able to optimally match the states of the analogously controlled uncertain system and those of the digitally controlled sampled-data uncertain system. Also, it gives less conservative results than those produced by the existing digital interval control law.

Appendix: Interval Analysis Preliminaries

Let the interval number $[a, b]$ be the set of $x_r \in \Re$ such that $a \leq x_r \leq b$. The arithmetic operations on intervals are defined as follows:

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \times [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

$$[a, b] \div [c, d] = [a, b] \times [(1/d), (1/c)] \quad \text{iff} \quad 0 \notin [c, d]$$

Note that the interval addition and multiplication are associative and commutative and the distributive law becomes subdistributive law, i.e.,

$$[a, b] \times ([c, d] + [e, f]) \subseteq [a, b] \times [c, d] + [a, b] \times [e, f]$$

Alternatively, the interval number x^I can be represented as

$$x^I = [\underline{x}, \bar{x}] := \{x_r \in \Re \mid \underline{x} \leq x_r \leq \bar{x}\} = [x_0 - \Delta x, x_0 + \Delta x]$$

where the nominal value $x_0 = (\underline{x} + \bar{x})/2$ and the uncertainty $\Delta x = (\bar{x} - \underline{x})/2$. The interval real number x_r in x^I is called a degenerate interval (real) number.

For all $n \times n$ interval real ($\Im\Re$) matrices $F^I = \{f_{ij}^I\} \in \Im\Re^{n\times n}$, with interval elements f_{ij}^I, and $G^I = \{g_{ij}^I\} \in \Im\Re^{n\times n}$, with interval elements g_{ij}^I for all i and j, the addition, subtraction, and multiplication operations can be written as

$$F^I \pm G^I = \left\{ f_{ij}^I \pm g_{ij}^I \right\} \in \Im\Re^{n\times n}, \qquad \text{all} \quad i, j$$

$$F^I G^I = \left\{ f_{ij}^I \right\}\left\{ g_{ij}^I \right\} = \left) \sum_{k=1}^{n} f_{ik}^I \times g_{kj}^I \right[\in \Im\Re^{n\times n}, \qquad \text{all} \quad i, j$$

The interval matrix addition and subtraction operations are associative and commutative. However, the interval matrix multiplication operation, in general, is not associative, and the distributive law becomes subdistributive law.

We use Hansen's method (see Ref. 25) to estimate the inverse of an interval matrix, as will be shown.

Let $A^I = \{a_{ij}^I\} \in \Im\Re^{n\times n}$ be an interval matrix and $A_0 = \{a_{0ij}\}$ an $n \times n$ real matrix (a degenerate interval matrix) with each entry $a_{0ij} = (\underline{a}_{ij} + \bar{a}_{ij})/2$. Thus, the inversion of the constant real matrix A_0 can be found by any matrix inversion algorithm. Also, let $E^I \in \Im\Re^{n\times n}$ be an interval error matrix given by

$$E^I = I_n - A^I A_0^{-1} = \left\{e_{ij}^I\right\} \in \Im\Re^{n\times n}, \qquad \text{all} \quad i, j$$

Define a norm of an interval matrix $F^I = \{f_{ij}^I\} \in \Im\Re^{n\times n}$ for all i and j as

$$\|F^I\| = \max_{1\le i\le n} \sum_{j=1}^{n} \max(|\underline{f}_{ij}|, |\bar{f}_{ij}|)$$

Notice that $\|F^I\|$ is an upper bound to the maximum row sum norm of any degenerate (real) matrix $F_0 = \{f_{0ij}\} \in F^I$, i.e.,

$$\|F^I\| \ge \|F_0\| = \max_{1\le i\le n} \sum_{j=1}^{n} |f_{0ij}|, \qquad \text{all} \quad i, j$$

If $\|E^I\| < 1$, the desired $(A^I)^{-1}$ satisfies

$$(A^I)^{-1} \subseteq A_0^{-1}\left[S_m^I + P_m^I\right] \in \Im\Re^{n\times n}$$

where

$$S_m^I = I_n + E^I\Big(I_n + E^I\Big(I_n + E^I\Big(I_n + E^I(\cdots)\Big)\Big)\Big)$$

to m sums and $P_m^I = \{P_{ij}^I\}$ is an $n \times n$ interval matrix with identical elements, each of which is the interval

$$p_{ij}^I = \left[\frac{-\|E^I\|^{m+1}}{1-\|E^I\|}, \frac{\|E^I\|^{m+1}}{1-\|E^I\|}\right], \qquad \text{all} \quad i, j$$

The use of Hansen's method gives a less conservative estimate of $(A^I)^{-1}$ than the direct use of any matrix inversion method with interval matrix operations. To reduce the inherent conservativeness of interval arithmetic operations, various improvements such as the use of nested form, centered form, etc., are proposed in Refs. 24–26 for interval arithmetic operations.

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